

Circumscribing constant-width bodies with polytopes

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Makeev conjectured that every constant-width body is inscribed in the dual difference body of a regular simplex. We prove that homologically, there are an odd number of such circumscribing bodies in dimension 3, and therefore geometrically there is at least one. We show that the homological answer is zero in higher dimensions, a result which is inconclusive for the geometric question. We also give a partial generalization involving affine circumscription of strictly convex bodies.

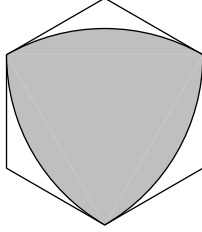


FIG. 1. A Rouleaux triangle inscribed in a regular hexagon

Any set of diameter 2 in \mathbb{R}^n is contained in a convex body of constant width 2. Consequently, if some polytope P circumscribes every convex body of constant width 2, it contains every set of diameter 2. For example, every constant-width body in two dimensions is inscribed in a regular hexagon (Figure 1). A conjecture of Makeev [4] generalizes this theorem to higher dimensions:

Conjecture 1 (V. V. Makeev). *Every constant width body in \mathbb{R}^n is inscribed in a polytope similar to D_n , the dual of the difference body of a regular simplex.*

The conjecture is motivated by the fact that D_n has $n(n+1)$ sides, the largest number possible for a polytope that has the circumscribing property [4]. Figure 2 illustrates D_3 , a standard rhombic dodecahedron.

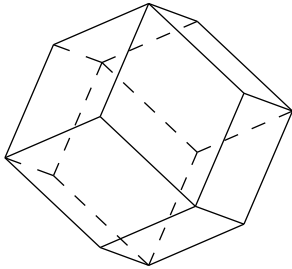


FIG. 2. The convex hull of D_3 , a rhombic dodecahedron

In this note, we will prove that every constant width body in \mathbb{R}^3 is circumscribed by an odd number of congruent copies of D_3 (in a homological sense), as is also the case in two dimensions. In particular, we prove Conjecture 1 for $n = 3$, a special case which was conjectured in 1974 by Chakerian. We also prove the following partial generalization:

Theorem 1. *Every strictly convex body in \mathbb{R}^3 is inscribed in a polyhedron which is affinely equivalent to the standard rhombic dodecahedron.*

It's not clear if the strict convexity condition is necessary.

In fact, Conjecture 1 and Theorem 1 can be generalized further: We can replace D_3 by the polyhedron

$$P = \{(x, y, z) : |x| \leq 1, |y| \leq 1, a|x| + a|y| + b|z| \leq \sqrt{2a^2 + b^2}\}$$

See Section VI.

All of these results are analogous to old results in two dimensions: Every convex body is circumscribed by an affinely regular hexagon and there are homologically an odd number of them [1]. Instead of a regular hexagon, we can take any centrally symmetric hexagon that circumscribes the unit circle.

Unfortunately, for $n \geq 4$, there are homologically zero circumscribing copies of D_n . However, this does not disprove Conjecture 1.

ACKNOWLEDGMENTS

The author has learned that Makeev [5] and independently Hausel, Makai, and Szucs [3] have obtained similar results.

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I. SUPPORT FUNCTIONS

We establish an equivalence between constant-width bodies and antisymmetric functions on the sphere.

Let K be a convex body in \mathbb{R}^n containing 0, the origin. For each unit vector v , let

$$f(v) = d(H_v, 0),$$

where H_v is the hyperplane which supports K , which is orthogonal to v , and which is on the same side of the origin as v . The function f is called the support function of K . The function

$$g(v) = f(v) - 1$$

be the adjusted support function of K .

Conversely, if g is any continuous function on the sphere $S^{n-1} \subset \mathbb{R}^n$ which is strictly less than 1, and if the spherical graph of

$$f(v) = 1/(g(v) + 1)$$

is convex, then g is the adjusted support function of some convex body K , namely the polar body of the graph of f . We will call such a function g pre-convex. Moreover, g is antisymmetric if and only if K has constant width 2. In conclusion, convex bodies in \mathbb{R}^n correspond to pre-convex functions on S^{n-1} and those that have constant width 2 correspond to antisymmetric pre-convex functions.

Proposition 1. *Let P be a polytope that circumscribes the sphere S^{n-1} and let T be the set of points at which it is tangent. Every convex body K (of constant width 2) is circumscribed by an isometric image of P if and only if every continuous (antisymmetric) function g agrees with a linear function on some isometric image of T .*

Proof. Let K be such a body and let g be its adjusted support function. The polytope P circumscribes K is equivalent to the statement that g vanishes identically on T . Translating K is equivalent to adding a linear function to g . This establishes the “if” direction of the proposition. It also establishes part of the “only if” direction, namely for pre-convex g rather than for arbitrary continuous g .

Consider the set X of all continuous g which agree with a linear function on some isometric image of T . This set is closed under multiplication by a scalar, and it is also a closed subset of the space of continuous functions on S^{n-1} taken with the Hausdorff topology. If X contains all pre-convex functions, then it must be the entire space of continuous functions, because every continuous function lies in the closure of the pre-convex functions in this double sense. (Any smooth function becomes pre-convex if multiplied by a sufficiently small constant and any continuous function can be approximated by smooth functions.) This completes the argument for the “only if” direction.

Both arguments also hold in the antisymmetric case. \square

Proposition 1 demonstrates that the circumscription problem for constant-width bodies belongs to a family of questions that includes the Knaster problem. This problem asks which finite families of points T on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ have the property that any continuous function from the sphere to \mathbb{R}^n is constant on an isometric image of T . The more general problem goes as follows: Given a finite set of points T on S^{d-1} and given a linear subspace L of the vector space of functions from T to \mathbb{R}^n , does every continuous function

$$f : S^{d-1} \rightarrow \mathbb{R}^n$$

admit an isometry R such that $f \circ R$ lies in L after restriction to T ? Even more generally, given any subspace V of finite codimension in the space of continuous functions on the sphere, does every continuous f admit an isometry R such

that $f \circ R \in V$? Of course the answer in general depends on V as well as d and n .

If the polytope P is the dual difference body D_n , then T is the set of vertices of the difference body of a regular simplex, also known as the root system A_n . In this case, Conjecture 1 is equivalent to the assertion that for every continuous, antisymmetric f on $S^{n-1} \subset \mathbb{R}^n$, there is a position of the root system A_n such that the restriction of f is linear.

II. TWO DIMENSIONS

The root system A_2 consists of six equally spaced points on the unit circle. Let C be the space of all isometric images T of A_2 . The set C is a topological circle. It has a natural 3-dimensional vector bundle F whose fiber at each $S \in C$ is the vector space of antisymmetric functions on T . If we divide this fiber by the linear functions on T , the result is a new vector bundle E on C . It is easy to check that the bundle E is a Möbius bundle.

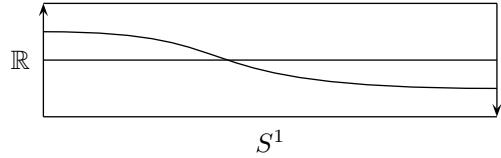


FIG. 3. A section of the Möbius bundle

If g is an antisymmetric, continuous function on the unit circle, it yields a section of F given by restricting g to each sextuplet T . In turn, one gets a section s of the bundle E . We wish to know whether the section s must have a zero. Since E is a Möbius bundle, this is true (Figure 3).

Thus we have proved that any constant-width body in the plane is circumscribed by a regular hexagon. The proof is actually just the traditional proof with some unconventional terminology. This terminology will be useful in the higher-dimensional cases.

III. THREE DIMENSIONS

We wish to show that every continuous, antisymmetric function on the 2-sphere agrees with a linear function on some isometric image of the root system A_3 , the vertices of a standard cuboctahedron (Figure 4). The set of such isometric images is a 3-manifold

$$M = \text{SO}(3)/\Gamma,$$

where Γ is the rotation group of A_3 (acting by right multiplication on $\text{SO}(3)$). The group Γ is the rotation group of the cube and is isomorphic to S_4 , the symmetric group on four letters. The manifold M has a 6-dimensional bundle F which at each point is the vector space of antisymmetric functions on

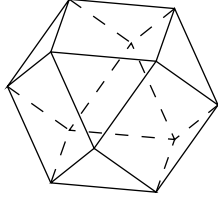


FIG. 4. A_3 forms a cuboctahedron

the corresponding image of A_3 . We quotient F by the linear functions to obtain a 3-dimensional bundle E .

We first rephrase the topological argument of the previous section in terms of characteristic classes of vector bundles [6]. An n -dimensional bundle B on an arbitrary topological space X (at least a reasonable one such as a CW complex) defines a characteristic cohomology class $\chi(B)$ called the Euler class. If X is a closed manifold, this class is dual to the homology class represented by the zero locus of a generic section of B . If B is orientable, then $\chi(B)$ is an element of the ordinary cohomology $H^n(X)$. But in general

$$\chi(B) \in H^n(X, \det(B)).$$

I.e., the Euler class lies in the cohomology of X in a twisted coefficient system, the determinant bundle of B . In our case, E is a non-orientable 3-plane bundle on the closed, orientable 3-manifold M . Therefore

$$\chi(E) \in H^3(M, \det(E)) \cong \mathbb{Z}/2.$$

In other words, the Euler class $\chi(E)$ is either 0 or 1, depending on whether a generic section has an even or odd number of zeroes.

Theorem 2. *The bundle E has a non-trivial Euler class:*

$$\chi(E) = 1 \in \mathbb{Z}/2.$$

Proof. There are two ways to argue this. The first way is by direct geometric construction. Consider the function xyz on S^2 . It produces a section s of E . The symmetry group of xyz , including antisymmetries, is the same group Γ ; thus, the section s has the same symmetries. The group Γ acts on the manifold M by means of symmetries that preserve or negate xyz but move some isometric image of A_3 . This is the *left* action of Γ on the coset space $M = \text{SO}(3)/\Gamma$; the quotient is the double coset space $\Gamma \backslash \text{SO}(3) / \Gamma$. The action has one fixed point (coming from the identity in $\text{SO}(3)$) and one orbit of size 3 (coming from a rotation by 45 degrees in $\text{SO}(3)$). All other orbits have even order. An elementary calculation shows that the fixed point is a transverse zero of the section s , while s is non-zero on the orbit of order 3. Thus the odd orbits make an odd contribution to the intersection between s and the zero section. The remaining zeroes of s , if there are any, lie on even-sized orbits and make an even contribution. Thus the Euler class of E is 1 and not 0.

The second way is by means of algebraic topology. Suppose that a vector bundle V on a space X lifts to a *trivialized* bundle \tilde{V} on some covering space \tilde{X} . Then V together with the choice of \tilde{V} is called a flat bundle. Both F and E are trivial if lifted to $\text{SO}(3)$, as well as flat on M , by construction. In general a flat bundle on a space X is described by some linear representation of the group of deck translations of \tilde{X} over X , assuming for simplicity that the covering is regular. In this case, the representation R of Γ that encodes E is simply the action of Γ on antisymmetric functions (modulo linear functions) on one copy of the A_3 root system. By writing down the character of this representation, or by writing down the representation explicitly, we can see that it is isomorphic to $V \otimes L$, where V is the 3-dimensional representation of $\text{SO}(3)$ restricted to Γ and L is the 1-dimensional representation of Γ coming from the sign homomorphism from $\Gamma = S_4$ to $\{\pm 1\}$. We can express this in terms of bundles with the equation

$$E \cong E_V \otimes E_L,$$

where E_V and E_L are the bundles defined by the representations V and L .

If a flat bundle X on a coset space G/H is given by a representation of H that is induced from G , it is a trivial bundle. For example, the bundle E_V is trivial for this reason. Thus the bundle E is actually three copies of the line bundle E_L . It is a general property of Euler classes that if X and Y are two bundles, the Euler class of the direct sum is the cup product of the Euler classes:

$$\chi(X \oplus Y) = \chi(X) \cup \chi(Y).$$

In this case we begin with the simpler Euler class

$$c = \chi(E_L) \in H^1(M, \mathbb{Z}/2)$$

from which we compute

$$\chi(E) = c \cup c \cup c.$$

We abbreviate $H^i(M, \mathbb{Z}/2)$ as just H^i . The cohomology group H^1 can be understood as the set of homomorphisms from $\pi_1(M)$ to $\mathbb{Z}/2$. In this case, all homomorphisms factor through Γ and $H^1 \cong \mathbb{Z}/2$. By this interpretation c is the same homomorphism as the one defining L , *i.e.*, the non-trivial one. By Poincaré duality, $H^2 \cong \mathbb{Z}/2$ as well, while $H^3 \cong \mathbb{Z}/2$ automatically because M is a closed 3-manifold. The cup product

$$\cup : H^1 \times H^2 \rightarrow H^3$$

is a non-degenerate pairing. To determine $\chi(E)$, the only question is whether $c \cup c$ is non-zero. In general, if X is a reasonable topological space and $x \in H^1(X, \mathbb{Z}/2)$ corresponds to a homomorphism from $\pi_1(X)$ to $\mathbb{Z}/2$, then $x \cup x$ vanishes if and only if the homomorphism lifts to $\mathbb{Z}/4$. One can check that the sign homomorphism of Γ does not lift, so $c \cup c$ is non-zero. Therefore the Euler class $\chi(E)$ does not vanish, as desired. \square

Let Γ_2 be the Sylow 2-subgroup of Γ and let

$$M_2 = \text{SO}(3)/\Gamma_2$$

be the corresponding covering space of M . Since the covering $M_2 \rightarrow M$ has odd degree, the lift of the bundle E to M_2 also has odd Euler class. This means that the theorem that every constant-width body is circumscribed by a D_3 generalizes to other polyhedra P with symmetry group Γ_2 , provided that the corresponding bundle E_P on M_2 is isomorphic to E , or that the corresponding representation is still R . For example, P can be any of the dodecahedra mentioned in the introduction.

IV. THE BAD NEWS

In any dimension n , there is a rotation group Γ which preserves the A_n root system and there is a manifold

$$M = \text{SO}(n)/\Gamma$$

of positions of the root system. The set of antisymmetric functions modulo linear functions is a flat bundle E on M whose dimension agrees with M . Let

$$d = n(n-1)/2$$

be the dimension of M .

If n is 0 or 1 modulo 4, the bundle E is orientable, and its Euler class is therefore an element of $H^d(M, \mathbb{Z})$, i.e., an integer, if an orientation is chosen. In general, the rational Euler class of a bundle X has a Chern-Weil formula, an expression in terms of the curvature of X . Since our bundle E is flat, this integral expression vanishes. The Euler class is therefore 0. Another way to argue this is that, as in 3 dimensions, E is a sum of line bundles. Negating one of the line bundles yields an orientation-reversing automorphism of E . The existence of such an automorphism tells us that the Euler class is its own negative.

If n is 2 or 3 modulo 4, the Euler class is an element of

$$H^d(M, \mathbb{Z}/2) \cong \mathbb{Z}/2.$$

We argue that for $n \geq 4$, this number also vanishes.

Proposition 2. *For $n \geq 4$, M admits a fixed-point free involution σ that extends to E .*

If we accept this proposition, we are done, since whatever $\chi(E)$ is on M/σ , it is an even multiple of it on M itself. It therefore vanishes modulo 2.

Proof. (Sketch) It suffices to find an involution g in $\text{SO}(n)$ that centralizes Γ but is not in Γ . For then the group Γ' generated by Γ and g would be a Cartesian product $\Gamma \times \mathbb{Z}/2$, the linear representation R would extend from Γ to Γ' , and the bundle E would descend from M to

$$M/g = \text{SO}(n)/\Gamma'.$$

The group of all isometries of a simplex in \mathbb{R}^n is the permutation group S_{n+1} . Adding central inversion, the full isometry group of D_n is

$$S_{n+1} \times \mathbb{Z}/2 \subset \text{O}(n).$$

The group Γ is an index 2 subgroup of this isometry group. The embedding of S_{n+1} in $\text{O}(n)$ is a linear representation which is almost the linear extension of the permutation representation on $n+1$ letters; the difference is that a trivial summand has been deleted. Let $S_{n+1;2}$ be the Sylow 2-subgroup of S_{n+1} . The action of $S_{n+1;2}$ on \mathbb{R}^n can be analyzed with arcane but standard computations. The property of this action that we need is that for $n \geq 4$, there are more representation endomorphisms in $\text{O}(n)$ (meaning isometries that commute with the action of $S_{n+1;2}$) than those provided by the center of $S_{n+1;2} \times \mathbb{Z}/2$ [2]. These extra endomorphisms include orientation-preserving involutions. The element g above can be any such involution. \square

The author also considered the natural conjecture that every constant-width body K in \mathbb{R}^4 is circumscribed by a regular cross polytope C (generalized octahedron). Since it has four fewer sides than the polytope D_4 , a 2-parameter family of copies of C circumscribes K if K is chosen generically. Unfortunately, another calculation shows that the set of such circumscribing polytopes is null-homologous in $\text{SO}(4)/\Gamma$, where Γ is the rotation group of C .

Finally, a constant-width body in \mathbb{R}^3 is inscribed in homologically zero regular dodecahedra. Chakerian has also asked whether there is always such a dodecahedron.

V. AFFINE CIRCUMSCRIPTION

Interestingly, the affine case of theorem 1 is a corollary of the constant-width case. For simplicity we begin with the argument in two dimensions. It is again an Euler class argument, except it is more complicated because the base space of the bundle is not compact. In this case a section of the bundle has a well-defined Euler class if it is proper, in the same sense that a map between non-compact spaces may be proper.

Let K be a convex body in the plane and let H be a regular hexagon whose inscribed circle has radius 1. Let

$$G = \text{GL}^+(2, \mathbb{R}) \ltimes \mathbb{R}^2$$

be the space of orientation-preserving affine transformations of the plane. There is a map Φ from G to \mathbb{R}^6 defined as follows: For a given affinity α , the coordinates of $\Phi(\alpha)$ are the distances from the lines containing the sides of H to $\alpha(K)$. If $\alpha(K)$ is on the same side of such a line as H is, the distance is taken to be negative, otherwise it is positive. Apparently Φ is continuous.

We wish to show that for a sufficiently small open neighborhood U of 0, $\Phi^{-1}(U)$ is bounded (contained in a compact subset of G), for then Φ has a well-defined degree. More precisely, we identify $\mathbb{R}^6 - U$ to a point to make the target of Φ a

ball, and we extend the domain to the one-point compactification of G . Then the degree of Φ is the degree of this modified map.

Lemma 1. *For a suitable U (independent of K) containing the origin, the region $\Phi^{-1}(U)$ is contained in a compact set.*

Proof. (Informally) We argue that if $\alpha \in G$ is sufficiently close to infinity, $\Phi(\alpha)$ is bounded away from 0. (Sufficiently close to infinity means sufficiently near the compactification point in the one-point compactification of G , or outside of a sufficiently large compact subset of G .) In general an element α may be close to infinity if the corresponding affine image $\alpha(K)$ has one of four properties: It may be translated far from H , it may be tiny, it may be enormous, or it may be highly anisotropic (needle-like). In the first three cases $\Phi(\alpha)$ is clearly bounded away from 0.

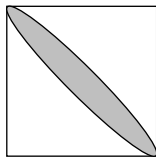


FIG. 5. A needle-like ellipse inscribed in a square

The last case is more subtle, particularly since the conclusion would not hold if H were a square rather than a hexagon (Figure 5). However, the smallest convex body inscribed in a regular hexagon is an equilateral triangle meeting three vertices. This follows from the more general fact that the smallest convex body inscribed in an arbitrary convex polygon is the convex hull of some of the vertices. (Such a body must touch each side and one of the endpoints of each side is always better than points in the middle.) If $\alpha(K)$ is so needle-like that its area is half of that of this triangle, then $\Phi(\alpha)$ is again bounded away from 0. \square

Since the set U in Lemma 1 is independent of K , and since K can be varied continuously, the degree of Φ is independent of K as well. Unfortunately it vanishes. However, the rotation group Γ of H acts on G and on \mathbb{R}^6 , and Φ is equivariant with respect to this action. Thus Φ represents a section of a bundle F on $W = G/\Gamma$ that also satisfies Lemma 1.

The section $\Phi : W \rightarrow F$ has an Euler class rather than a degree. To compute it we take K to be the unit circle. The zero locus of Φ is then $M = \text{SO}(2)/\Gamma$, the manifold that appears in the constant-width case. Moreover, Φ is transverse to the zero section of F in the directions normal to M . These directions are characterized by affinities whose matrices are symmetric, *i.e.*, by stretching or squeezing K along orthogonal axes. The derivative of such a motion is radially a homogeneous quadratic function on the boundary of the circle K . The key fact to check is that a homogeneous quadratic function is determined by its values on A_2 , the tangencies of the hexagon H . In other words, the derivative of Φ here is essentially restriction to A_2 , a linear transformation which is nonsingular for homogeneous quadratic functions. If we quotient F on M

by the image under Φ of the normal bundle NM of M , we are left with the bundle E on M considered previously. Thus the Euler class of Φ on W equals the Euler class of E on M , namely $1 \in \mathbb{Z}/2$.

This argument generalizes verbatim to three dimensions, except that unfortunately Lemma 1 no longer holds. Among closed convex sets inscribed in the rhombic dodecahedron D_3 , a square, which has volume zero, has the least volume (Figure 6). The square is the unique minimum up to isometry. If K is strictly convex, its affine image $\alpha(K)$ is bounded away from a square, and therefore $\Phi(\alpha)$ is again bounded away from 0 for α sufficiently close to infinity.

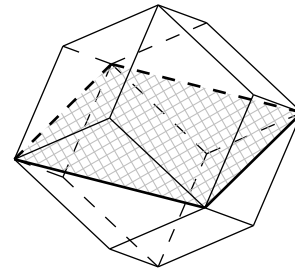


FIG. 6. A square inscribed in a rhombic dodecahedron

Thus in three dimensions the Euler class of Φ is well-defined when K is strictly convex. Moreover, a finite path $\{K_t\}_{t \in [0,1]}$ of strictly convex bodies is strictly convex in a uniform fashion by compactness. Therefore the Euler class of Φ does not change along such a path. For every strictly convex K it must always equal its value when K is a round sphere, namely $1 \in \mathbb{Z}/2$.

VI. ODDS AND ENDS

Following the computations of Section IV, we did not really need the full symmetry group of the rhombic dodecahedron D_3 full symmetry group, but only its Sylow 2-subgroup Γ_2 and the way that this subgroup permutes its faces. Because if we lift the bundle E of Section III to an odd-order covering of M , its Euler class remains non-zero. Thus the argument applies to any other polytope which is symmetric under Γ_2 , whose faces are permuted by Γ_2 in the same way, which is centrally symmetric, and which circumscribes the sphere. In particular the results hold for the polytope P described in the introduction.

It would be interesting if there were a convex body K which does not affinely inscribe in a rhombic dodecahedron. We can obtain some information about such a K from the arguments of Section V. It would necessarily affinely project onto a square. Given any sequence of strictly convex bodies

$$K_1, K_2, \dots \rightarrow K,$$

their affine inscriptions in D_3 would necessarily converge to an inscribed square. Affine circumscriptions of D_3 around

each K_n would converge to an infinite parallelogram prism circumscribing K , and K would meet all four edges of this prism. Otherwise some subsequence of the affine images of D_3 would converge to an affine image circumscribing K .

- [1] B. Grünbaum. On inscribing and circumscribing hexagons. *Colloq. Math.*, 17:99–101, 1967.
- [2] R. Guralnick, 1997. private communication.
- [3] T. Hausel, E. Makai Jr., and A. Szucs. Inscribing cubes and covering by rhombic dodecahedra via equivariant topology, math.MG/9906066.
- [4] V. V. Makeev. Inscribed and circumscribed polygons of a convex body. *Mat. Zametki*, 55:128–130, 1994.
- [5] V. V. Makeev. On affine images of a rhombo-dodecahedron circumscribed about a three-dimensional convex body. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 246:191–195, 1997.
- [6] J. W. Milnor and J. D. Stasheff. *Characteristic Classes*. Number 74 in Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1974.